# Section 5.5

Wednesday, April 8, 2020 6:27 PM

## Antiderivatives

A function $F(x)$ is called **an antiderivative for $f(x)$ on $I$** if $F'(x) = f(x)$ for all $x$ in $I$, where $I$ is an open interval.

Given diff. functions $F(x)$ and $G(x)$, we have that $F(x)$ and $G(x)$ are antiderivatives of the same function if and only if $F(x) - G(x) = C$ for some constant $C$, on each interval in their domain.

Given a function $f$, we will define **the indefinite integral** of $f$ as the class of all antiderivatives of $f$ on the domain of $f$ and denote this by

$$\int f(x)\,dx = F(x) + C$$

where $F$ is an antiderivative of $f$ on the domain of $f$.

When we write $\int f(x)\,dx = F(x) + C$, it means that any antiderivative of $f$ differs from $F$ by a constant on **each** interval that makes up the domain of $f$.

$$\int \frac{-1}{x^2}\,dx = \frac{1}{x} + C$$

$$\boxed{F(x) = \begin{cases} \frac{1}{x} + 1 & \text{if } x > 0 \\ \frac{1}{x} - 7 & \text{if } x < 0 \end{cases} \qquad F'(x) = \frac{-1}{x^2}}$$

## Some basic indefinite integrals

- $\int x^p\,dx = \dfrac{x^{p+1}}{p+1} + C$ for $p \neq -1$
- $\int \frac{1}{x}\,dx = \ln|x| + C$
- $\int \sin x\,dx = -\cos x + C$
- $\int \cos x\,dx = \sin x + C$
- $\int e^x\,dx = e^x + C$
- $\int \sec^2 x = \tan x + C$
- $\int \sec x \tan x = \sec x + C$
- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$
- $\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$

# The Fundamental Theorem of Calculus

Let $f$ be a continuous function on an interval $I$ containing the point $a$.

- **(Part I)** Then the function $F(x) = \int_a^x f(t)\,dt$ is differentiable on $I$ and moreover,

$$F'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

- **(Part II)** For any $b$ in $I$, $\int_a^b f(t)\,dt = G(b) - G(a)$ for any antiderivative $G$ of $f$ over $I$.

**Proof:** (Part I) By defn, we have

$$\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt$$

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

$$= \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

the average value of $f$ on $[x, x+h]$ and is equal to $f(c_h)$ for some $c_h$ between $x$ and $x+h$

$$= \lim_{h\to 0} f(c_h) = f\left(\lim_{h\to 0} c_h\right) = f(x)$$

**(Part II)** Let $G$ be any antiderivative of $f$ on $I$. Then, since $F(x) = \int_a^x f(t)\,dt$ is also an antiderivative of $f$ on $I$, we get $G(x) - F(x) = c$ for some $c$. Plugging in $x = a$ gives $G(a) - F(a) = G(a) = c$. Thus we get

$$F(x) = G(x) - G(a)$$

Now we plug in $x = b$ and get

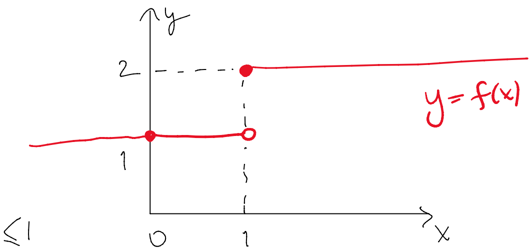
$$\int_a^b f(t)\,dt = G(b) - G(a) = G(x)\Big|_a^b$$

$\blacksquare$

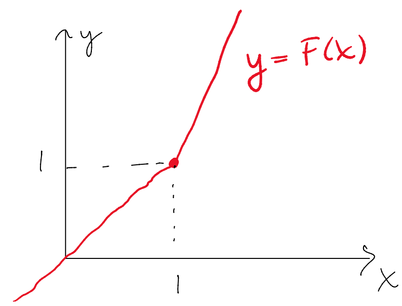
**Notation:** We shall denote $F(x)\Big|_a^b = F(b) - F(a)$.

**WARNING:** The assumption that $f$ is continuous in FTC is crucial.

- $f(x) = \begin{cases} 1 & \text{if } x < 1 \\ 2 & \text{if } 1 \le x \end{cases}$



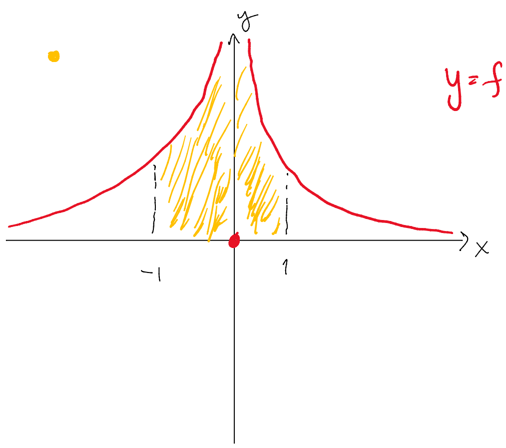

$$F(x) = \int_0^x f(t)\,dt = \begin{cases} x & \text{if } x \le 1 \\ 1 + 2(x-1) & \text{if } 1 < x \end{cases}$$




$F'(1)$ does not exist so

is not equal to $f(1) = 2$

$$F'(1) \neq f(1)$$

- 

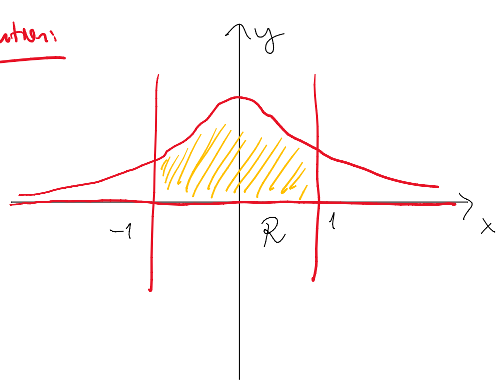

$$y = f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

The function $\frac{-1}{x}$ is an antiderivative of $\frac{1}{x^2}$ on $(0,\infty)$ and $(-\infty, 0)$.

$$\int_{-1}^{1} f(x)\,dx \neq \left.\frac{-1}{x}\right|_{-1}^{1} = \left(\frac{-1}{1}\right) - \left(\frac{-1}{-1}\right) = -1 - 1 = -2$$

**Example:** Find the area of the region bounded by $y = \frac{1}{1+x^2}$, $x = -1$, $x = 1$, $y = 0$.

**Solution:**




The area of $R$ is

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx \overset{\text{FTC II}}{=} \arctan(x)\Big|_{-1}^{1} = \arctan(1) - \arctan(-1)$$

$$= \frac{\pi}{4} - \frac{-\pi}{4} = \frac{\pi}{2}$$

**Example:** Find the average value of $f(x) = e^x + \sin x$ over $[0, \pi]$.

**Solution:** The average value of $f(x)$ over $[0, \pi]$ is

$$\frac{\int_0^\pi f(x)dx}{\pi - 0} = \frac{\int_0^\pi e^x + \sin x \, dx}{\pi} \overset{\text{FTC II}}{=} \frac{(e^x - \cos x)\Big|_0^\pi}{\pi} = \frac{(e^\pi - \cos\pi) - (e^0 - 0)}{\pi}$$

$$= \frac{e^\pi + 1 - 1}{\pi} = \frac{e^\pi}{\pi}$$

**Example:** Find the inflection points of $F(x) = \int_0^x e^{t^2} dt$.

**Solution:** By FTC I, we have $F'(x) = e^{x^2}$. So $F''(x) = e^{x^2} \cdot 2x$.

| $x$ | $-\infty$ | $0$ | $\infty$ |
|---|---|---|---|
| $F''(x)$ | $-$ | $0$ | $+$ |
| $F(x)$ | $\frown$ | inflection point | $\smile$ |

$F''(x) = e^{x^2} 2x = 0 \Rightarrow x = 0$

Thus $F$ has an inflection point at $x = 0$.

**Example:** Find $\frac{d}{dx}\left(\int_{\sin x}^1 \cos t \, dt\right)$

**Solution:**

$$\frac{d}{dx}\left(\int_{\sin x}^1 \cos t \, dt\right) = \frac{d}{dx}\left(-\underbrace{\int_1^{\sin x} \cos t \, dt}_{F(\sin x)}\right) = -F'(\sin x) \cdot (\sin x)'$$

$$= -\cos(\sin x) \cdot \cos x$$

$$\boxed{\text{Let } F(x) = \int_1^x \cos(t)\,dt. \text{ Then by FTC I, } F'(x) = \cos x. \text{ So}}$$

**Example:** Find $\frac{d}{dx}\left(\int_{x^2+1}^{x^4+1} \sqrt{1+\ln(t)}\, dt\right)$

**Solution:**

$$\frac{d}{dx}\left(\int_{x^2+1}^{x^4+1} \sqrt{1+\ln(t)}\, dt\right) = \frac{d}{dx}\left(\int_{x^2+1}^{2} \sqrt{1+\ln(t)}\, dt + \int_2^{x^4+1} \sqrt{1+\ln(t)}\, dt\right)$$

$$= \frac{d}{dx}\left( - \int_2^{x^2+1} \sqrt{1+\ln(t)}\,dt + \int_2^{x^4+1} \sqrt{1+\ln(t)}\,dt \right)$$

$$\boxed{\begin{aligned} F(x) &= \int_2^x \sqrt{1+\ln(t)}\,dt \\ F'(x) &= \sqrt{1+\ln(x)} \end{aligned}}$$

$$= \frac{d}{dx}\left( - F(x^2+1) + F(x^4+1) \right)$$

$$= -F'(x^2+1) \cdot 2x + F'(x^4+1) \cdot 4 \cdot x^3$$

$$= -\sqrt{1+\ln(1+x^2)} \cdot 2x + \sqrt{1+\ln(x^4+1)} \cdot 4 \cdot x^3$$

**Example:** Find $\displaystyle \lim_{x \to 0} \frac{\int_0^x \sin(t^4)\,dt}{\int_0^x (\cos(t^2)-1)\,dt}$

**Solution:**

$$\lim_{x \to 0} \frac{\overbrace{\int_0^x \sin(t^4)\,dt}^{F(x)}}{\underbrace{\int_0^x (\cos(t^2)-1)\,dt}_{G(x)}} \left(\frac{0}{0}\right) \overset{L'H}{} \lim_{x \to 0} \frac{F'(x)}{G'(x)} \overset{FTC I}{=} \lim_{x \to 0} \frac{\sin(x^4)}{\cos(x^2)-1} \left(\frac{0}{0}\right) \overset{L'H}{}$$

$$\lim_{x \to 0} \frac{\cos(x^4) \cdot \cancel{4x^3}^{\,2x^2}}{-\sin(x^2) \cdot \cancel{2x}} = \lim_{x \to 0} - \frac{\cancel{\cos(x^4)}^{\,1}}{\frac{\cancel{\sin(x^2)}^{\,1}}{\cancel{x^2}}} \cdot 2 = -2$$

as $\displaystyle \lim_{t \to 0} \frac{\sin(t)}{t} = 1$